MR1865567 (2002m:14009) 14E15 14B05 14J17 32S45
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The Nash sheaf of a complete resolution. (English summary)
Manuscripta Math. **106** (2001), *no. 2,* 249–270.

Let V be a three-dimensional algebraic variety over $\mathbb{C}$ with isolated singular point v, and let U be a neighborhood of v in V with an embedding $(U,v)\subseteq(\mathbb{C}^N,0)$. Let $\pi\colon\widetilde{U}\to U$ be a resolution of the singularity v in V with exceptional divisor E (i.e. $E=\pi^{-1}(v)$) and let m be the sheaf-theoretic inverse image of the maximal ideal sheaf m_v of v.

If $\mathcal{N}$ is a sheaf on a blow-up $\pi\colon\widetilde{U}\to U$, $\mathcal{N}$ is called a generalized Nash sheaf if $\widetilde{U}$ factors through the Nash blow-up $\widehat{\pi}\colon\widehat{U}\to U$ and if $\mathcal{N}$ is the pullback of the Nash sheaf on $\widehat{U}$. When $\pi\colon\widetilde{U}\to U$ is a resolution of the isolated singularity of V factoring through the Nash blow-up of U, there are the inclusion $\mathcal{N}_{\widetilde{U}}\hookrightarrow\Omega^1_{\widetilde{U}}$ obtained by the construction of the Nash sheaf and the inclusion $\Omega^1_{\widetilde{U}}\hookrightarrow\Omega^1_{\widetilde{U}}(\log E)$ of the sheaf of 1-forms into the sheaf of 1-forms with logarithmic singularities along the exceptional divisor E. The second Fitting ideal sheaf $\mathcal{F}$ of the cokernel sheaf $\Omega^1_{\widetilde{U}}(\log E)/\mathcal{N}_{\widetilde{U}}$ is the sheaf of ideals generated by the 2×2 minors of the matrix associated to the inclusion of the generalized Nash sheaf $\mathcal{N}_{\widetilde{U}}$ into $\Omega^1_{\widetilde{U}}(\log E)$.

Under these conditions, the author says that the resolution $\pi\colon\widetilde{U}\to U$ is complete if m is locally principal, $\mathcal{N}$ is locally free and $\mathcal{F}$ is locally principal over $\widetilde{U}$.

The principal results of the paper are the following: Theorem 1. Given a three-dimensional complex algebraic variety V with isolated singular point v and a sufficiently small affine neighborhood U of v in V, there exists a complete resolution $\pi\colon(\widetilde{U},E)\to(U,v)$.

The second theorem proves that the complete resolutions are "complete" in the sense that no further blow-ups will be necessary to construct Hsiang-Pati coordinates in the sense of [W. C. Hsiang and V. Pati, Invent. Math. **81** (1985), no. 3, 395–412; MR0807064] near any point e in the exceptional divisor E. Namely, the result is the following: Theorem 2. Given a three-dimensional complex algebraic variety V with isolated singular point v, let U be an affine neighborhood of v in V with complete resolution $\pi\colon(\widetilde{U},E)\to(U,v)$. Choose a point $e\in E$ and an analytic neighborhood W of e in $\widetilde{U}$. Then there exists a set of divisor coordinates $\{u,v,w\}$ on W such that the Nash sheaf $\mathcal{N}$ is locally generated by the differentials $d\phi,d\psi,d\rho$ of monomial functions of the form $\phi=u^{m_1}v^{m_2}w^{m_3}$, $\psi=u^{n_1}v^{n_2}w^{n_3}$, $\rho=u^{p_1}v^{p_2}w^{p_3}$ whose exponents $\{(m_1,m_2,m_3),(n_1,n_2,n_3),(p_1,p_2,p_3)\}$ are a Hsiang-Pati ordered set.

The last result is a generalization of the above result in the following sense: For a complete resolution $\pi\colon(\widetilde{U},E)\to(U,v)$ and linear functions j,k and l on U, the triple $\phi:=j\circ\pi$, $\psi:=k\circ\pi$ and $\rho:=l\circ\pi$ of functions on $\widetilde{U}$ is called Nash-minimal with respect to $e\in E$ with analytic neighborhood W if (1) ϕ is a generator of $m(W)$; (2) $\{d\phi,d\psi,d\rho\}$ is a generating set for $\mathcal{N}(W)$; and (3) $d\phi$ and $d\psi$ are minimal elements of $\bigwedge^2\mathcal{N}(W)$. The author proves: Theorem 3. Given a three-dimensional complex algebraic variety V with isolated singular point v, let V be an affine neighborhood of v in V with complete resolution $\pi\colon(\widetilde{U},E)\to(U,v)$. Choose a point $e\in E$ and an analytic neighborhood W of e in $\widetilde{U}$. Let $\{\phi,\psi,\rho\}$ be a Nash-minimal set of functions with respect to the point e. Then there exist a set of divisor coordinates $\{u,v,w\}$ on W and a set $\{(m_1,m_2,m_3),(n_1,n_2,n_3),(p_1,p_2,p_3)\}$ of Hsiang-Pati ordered integers such that: (1) $\phi = u^{m_1}v^{m_2}w^{m_3}$; (2) $\psi = S + \psi'$, where (a) $S = \sum s_l \phi^{\epsilon_l}$ with each $\epsilon_l \geq 1$ and rational; and (b) $\psi' = u^{n_1}v^{n_2}w^{n_3}$; (3) $\rho = T + \rho'$, where (a) $T = \sum t_l \phi^{\delta_l}(\psi')^{\tau_l}$ with $\delta_l \geq 1$ when $\tau = 0$ and $\delta_l m_1 + \tau_l n_1 \geq n_1$ when $\tau \neq 0$ (and similarly for m_2 and n_2 and for m_3 and n_3 if e is a triple point, or for m_2 and n_2 if e is a double point) and (b) $\phi' = u^{p_1}v^{p_2}w^{p_3}$.

The functions $\{\phi, \psi, \rho\}$ on $\widetilde{U}$ are called Hsiang-Pati coordinates, which can be generalized to the n-dimensional case in the author's work "where in the more general case a complete resolution will factor through the blow-ups of a 'series' of Fitting ideal sheaves". Finally, the paper under review is based on the Ph.D. thesis of the author ["Monomial generators for Nash sheaf of a complete resolution", Duke Univ., Durham, NC, 2000] which had W. L. Pardon as thesis advisor. In this context, the results included in this interesting paper are a generalization of the results of [W. L. Pardon and M. A. Stern, J. Reine Angew. Math. **533** (2001), 55–80; MR1823864].

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.